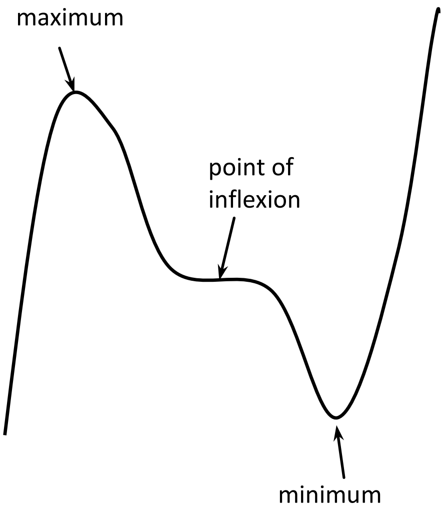
Stationary points

In this activity you will use differentiation to find stationary points then sketch curves.

Information sheet

There are 3 types of stationary points:
maximum points,
minimum points, and
points of inflexion.



Think about…

What happens to the gradient as the curve passes through:

- the maximum point?
- the minimum point?
- a point of inflexion?

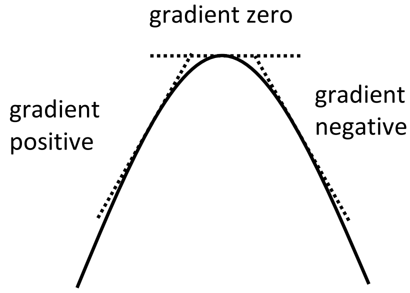
Maximum points

Just before a **maximum point** the gradient is positive.
At the maximum point the gradient is zero, and
just after the maximum point it is negative.

The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing,

so the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x is negative.

This means $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative at the maximum point.



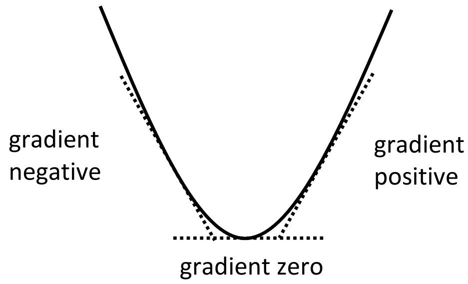
Minimum points

Just before a **minimum point** the gradient is negative,
at the minimum the gradient is zero, and
just after the minimum point it is positive.

The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing, so

the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x is positive.

This means $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is positive.

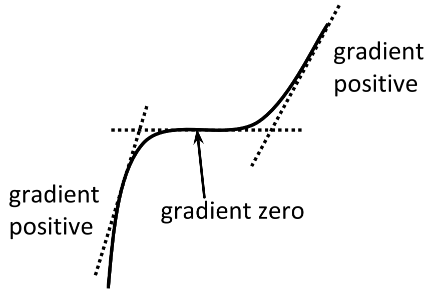


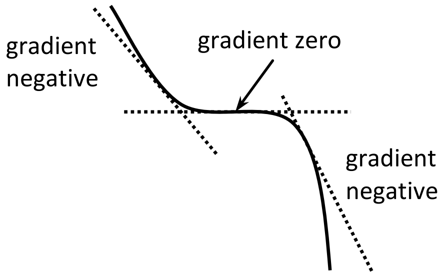
Points of inflexion

At some points like those shown

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$.

These points are called **points of inflexion**.





Note that $\frac{d^2y}{dx^2}$ is also zero at some maximum and minimum points.

To find the type of stationary point, consider the gradient at each side of it.

To sketch a curve

Find the stationary point(s)

Find an expression for $\frac{dy}{dx}$ and put it equal to 0,

then solve the resulting equation to find the x coordinate(s) of the stationary point(s).

Find $\frac{d^2y}{dx^2}$ and substitute each value of x to find the kind of stationary point(s).

(+ suggests a minimum, – a maximum,
0 could be either of these or a point of inflexion)

Use the curve's equation to find the y coordinate(s) of the stationary point(s).

Find the point(s) where the curve meets the axes

Substitute $x = 0$ in the curve's equation to find the y coordinate of the point where the curve meets the y axis.

Substitute $y = 0$ in the curve's equation. If possible, solve the equation to find the x coordinate(s) of the point(s) where the curve meets the x axis.

Sketch the curve, then use a graphic calculator to check.

Example A To sketch the curve $y = 5 + 4x - x^2$

$$\frac{dy}{dx} = 4 - 2x \text{..........(1)}$$

At stationary points $\frac{dy}{dx} = 0$

This gives $2x = 4$ so $x = 2$

From (1) $\frac{d^2y}{dx^2} = -2$ suggesting a maximum.

Substituting $x = 2$ into $y = 5 + 4x - x^2$ gives:

$y = 5 + 8 - 4 = 9$ so **(2, 9)** is the **maximum point**

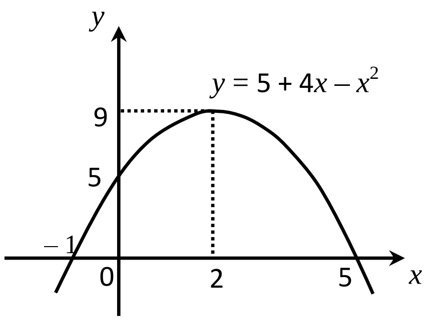
When $x = 0$ $y = 5$

When $y = 0$ $5 + 4x - x^2 = 0$

$(5 - x)(1 + x) = 0$

so $x = 5$ or -1

Curve **crosses the axes** at **(– 1, 0)** and **(5, 0)**



Example B To sketch the curve $y = 2 + 3x^2 - x^3$

$$\frac{dy}{dx} = 6x - 3x^2 \text{.......(1)}$$

At stationary points $\frac{dy}{dx} = 0$

This means $6x - 3x^2 = 0$

Factorising gives $3x(2 - x) = 0$

with solutions $x = 0$ or $x = 2$

From (1) $\frac{d^2y}{dx^2} = 6 - 6x$

which is positive when $x = 0$ and negative when $x = 2$.

Substituting the values of x into $y = 2 + 3x^2 - x^3$:

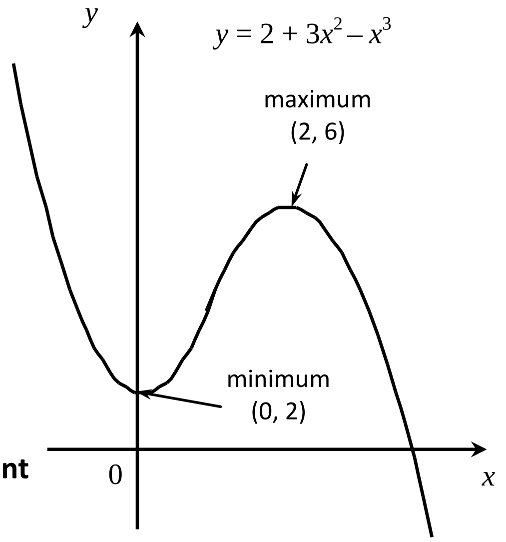
$x = 0$ gives $y = 2$ and $x = 2$ gives $y = 6$

(0, 2) is a **minimum point** and **(2, 6)** is a **maximum point**

When $x = 0$ $y = 2$

When $y = 0$ $2 + 3x^2 - x^3 = 0$

Solving such cubic equations is difficult and not necessary.
It is possible to sketch the curve using just the stationary points and the fact that it **crosses the** y **axis** at **(0, 2).**



Note

Use a graphic calculator to check the sketch. If you wish, you can use the trace function to find the x coordinate of the point where the curve crosses the x axis. In this case the curve crosses the x axis at approximately (3.2, 0).

Example C To sketch the curve $y = x^4 - 4$

$$\frac{dy}{dx} = 4x^3 \text{..........}(1)$$

At stationary points $\frac{dy}{dx} = 0$

This gives $4x^3 = 0$ so $x = 0$ and $y = -4$

From (1) $\frac{d^2y}{dx^2} = 12x^2 = 0$ when $x = 0$

When $x = 0$ $y = -4$

When $y = 0$ $x^4 - 4 = 0$

$x^4 = 4$

so $x^2 = 2$ and $x = \pm\sqrt{2}$

Curve crosses the axes at **(0, – 4)**, **(– √2, 0)** and **(√2, 0)**

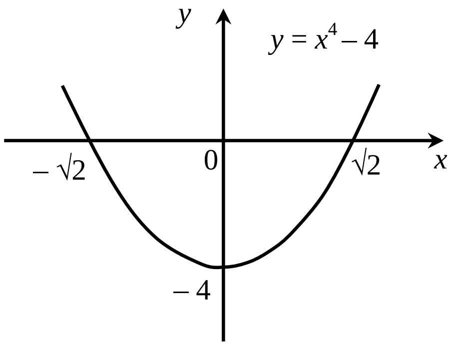
In this case the stationary point could be a maximum, minimum, or point of inflexion.

To find out which, consider the gradient before and after $x = 0$.

When x is negative $\frac{dy}{dx} = 4x^3$ is negative

When x is positive $\frac{dy}{dx}$ is positive

so **(0, – 4)** is a **minimum point**



Think about…

- Why is $\frac{d^2y}{dx^2}$ negative for a maximum point?
- How can you tell where the curve crosses the y axis and the x axis?
- How can you tell what the function does as $x \to \pm\infty$?

Try these

1 For each of the curves whose equations are given below:

- find each stationary point and what type it is
- find the coordinates of the point(s) where the curve meets the x and y axes
- sketch the curve
- check by sketching the curve on your graphic calculator.

a $y = x^2 - 4x$ **b** $y = x^2 - 6x + 5$

c $y = x^2 + 2x - 8$ **d** $y = 16 - x^2$

e $y = 6x - x^2$ **f** $y = 1 - x - 2x^2$

g $y = x^3 - 3x^2$ **h** $y = 16 - x^4$

i $y = x^3 - 3x$ **j** $y = x^3 + 1$

2 For each of the curves whose equations are given below:

- find each stationary point and what type it is
- find the coordinates of the point where the curve meets the y axis
- sketch the curve
- check by sketching the curve on your graphic calculator.

a $y = x^3 + 3x^2 - 9x + 6$ **b** $y = 2x^3 - 3x^2 - 12x + 4$

c $y = x^3 - 3x - 5$ **d** $y = 60x + 3x^2 - 4x^3$

e $y = x^4 - 2x^2 + 3$ **f** $y = 3 + 4x - x^4$

Reflect on your work

- How does finding stationary points help you to sketch a curve?
- Is it possible to know how many stationary points there are by just looking at the function?
- What other information is it useful to find before you try to sketch a curve?